Pierre Gy‘s Approach to Model the Liberation Factor

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ABSTRACT

Several approaches have been suggested by various authors to model the liberation factor when calculating the variance of the Fundamental Sampling Error. One of these approaches is the result of a thorough theoretical analysis and derivation authored by Dr. Pierre Gy as early as 1967.

Another approach, empirical in nature, models the liberation factor as a function of the top size of the fragments and the liberation size of the constituent of interest.

Both approaches have their respective merits. However, the empirical approach, as popular as it may be, is tampering with the theoretical integrity of Pierre Gy’s formula and may become highly misleading depending on how it is applied.

This paper emphasizes the superiority of the theoretical approach, explains the reasons why it should be the only valid approach, irrespective of some inconveniences and limitations, and finally suggests ways to eliminate the need for a liberation factor.

1. The Historical Theoretical Approach Created by Pierre Gy

The earliest publication of Dr. Pierre M. Gy is the special edition of “L’Industrie Minerale” (1967) in which the concept of the Fundamental Sampling Error (FSE) is brilliantly introduced. It is shown that FSE is the smallest possible sampling uncertainty if, and only if, for a given sample mass the conditions of equiprobability have been perfectly respected.

$m(FSE) \approx 0$ in a first order approximation [1]

It is shown that the variance of FSE can be expressed as follows:

$$s^2(FSE) = \frac{1-P}{P} \sum_i \left[\frac{(a_i - a_L)^2}{a_L^2} \cdot \frac{M_i^2}{M_L^2}\right] = \frac{1-P}{P \cdot N_{Fi}} CH_L \qquad [2]$$

Where P is the sampling probability, a_i is the content of the constituent of interest in any fragment, a_L the average content of the lot to sample, M_i the mass of any fragment, M_L the mass of the lot and CH_L the Constitution Heterogeneity of the lot.

We don’t want to count the number of fragments in the lot. It is easy to overcome this difficulty by multiplying CH_L by a constant factor such as the average weight of a fragment which is by definition $\overline{M_i} = \frac{M_L}{N_{Fi}}$.

Then we have to define a new term called the *Constant Factor of Constitution Heterogeneity* IH_L also called *Intrinsic Heterogeneity* by some authors.

$$IH_L = \sum_i \frac{(a_i - a_L)^2}{a_L^2} \cdot \frac{M_i^2}{M_L} \qquad [3]$$

Therefore, the following important relation can be obtained:

$$s^2(FSE) = \frac{1-P}{P \cdot M_L} IH_L \qquad [4]$$

From this point several pragmatic formulas can be derived which have their own domain of application and limitations. For the record: there is no such thing as the Pierre Gy’s *magic* formula as is the perception given by many people around the world who are not familiar with the subtleties of his valuable work.

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2. A pragmatic formula for a parametric approach

$$IH_L = \sum_\alpha v_\alpha \sum_\beta \rho_\beta \frac{(a_{\alpha\beta} - a_L)^2}{a_L^2} \cdot \frac{M_{L\alpha\beta}}{M_L} \qquad [5]$$

The following, well-known formula is complete and still relatively simple, where α refers to a given size fraction, and β refers to a given density fraction.
The theoretical foundation of a quick approach and its limitations is the most appropriate start. Let's begin with the approximate formula used from the development of Gy's parametric approach:

$$IH_L = X \cdot Y = \left[\frac{\sum_\alpha v_\alpha \cdot M_{L\alpha}}{M_L}\right] \left[\sum_\beta \rho_\beta \frac{(a_\beta - a_L)^2}{a_L^2} \cdot \frac{M_{L\beta}}{M_L}\right] \qquad [6]$$

This simplified equation is based on two important assumptions:

1. Experience shows that the content $a_{a\beta}$ of a constituent of interest usually varies much more from one density fraction ß to the next than from one size fraction a to the next; therefore all the values of $a_{a\beta}$ obtained in a size-density heterogeneity experiment may be replaced by the average content a_β of the corresponding density fraction L_β. This assumption is almost always true.
2. The study of a large number of real cases shows that in a size-density heterogeneity experiment the proportions $\frac{M_{L\alpha\beta}}{M_{L\beta}}$ usually varies little from one density fraction to the next; therefore, we may assume that all values $\frac{M_{L\alpha\beta}}{M_{L\beta}}$ can be replaced by their average $\frac{M_{L\alpha}}{M_L}$. This assumption may become debatable in some rare cases.

The *X* term is relative to the size fractions and leads to the shape factor and particle size distribution factor, while the term *Y*, relative to the density fractions is the one of interest in our present analysis. We know *Y* can be expressed as follows:

$$Y = \sum_\beta \rho_\beta \frac{(a_\beta - a_L)^2 M_{L\beta}}{a_L^2 \cdot M_L} \qquad [7]$$

2.1 The mineralogical factor

Y reaches a maximum when the constituent of interest is completely liberated. This maximum is defined as the mineralogical factor *c*. If the material is made of only two liberated constituents, for example the gangue and the mineral of interest, then the density fraction containing the pure liberated mineral has a density ρ_M, while the gangue fraction has a density ρ_g. Then, it follows that for the fraction containing the mineral of interest the content of the mineral is $a_\beta = a_M = 1$; then for the gangue fraction $a_\beta = a_g = 0$; also the ratio $\frac{M_M}{M_L} = a_L$ and the ratio $\frac{M_g}{M_L} = (1 - a_L)$.

Transposing these values in equation [7]:

$$Y = c = \rho_M \frac{(1 - a_L)^2 M_M}{a_L^2 \cdot M_L} + \rho_g \frac{(0 - a_L)^2 M_g}{a_L^2 \cdot M_L}$$

After simplifications we obtain:

$$c = \rho_M \frac{(1 - a_L)^2}{a_L} + \rho_g (1 - a_L) \qquad [8]$$

2.2 The liberation factor

Let's make several hypotheses which must be kept in mind to understand the limitations of the following recommended methods. Let's also assume we are interested in the copper content of a lot to be sampled.

First hypothesis: Following an analytical investigation, we suppose that the maximum copper content a_{max} of the coarsest fragments of the lot is known.

Second hypothesis: We suppose that all size fractions have roughly the same copper content a_L, or at least they are within the same order of magnitude.

Third hypothesis: We suppose that inside each fraction all of the copper is located in a sub fraction of copper content a_{max}, density ρ_R, and relative weight

$$\frac{M}{M_L} = \frac{a_L}{a_{\max}}, \qquad [9]$$

while the remainder of the size fraction of density ρ_g, and relative weight $1 - \left(\frac{a_L}{a_{\max}}\right)$ does not contain any copper or very little.

Then we can rewrite equation [7] as follows:

$$Y = \rho_R \frac{(a_{\max} - a_L)^2 a_L}{a_L^2 a_{\max}} + \rho_g \frac{(0 - a_L)^2}{a_L^2} \left(1 - \frac{a_L}{a_{\max}}\right) \qquad [10]$$

which leads after simplification to:

$$Y = \rho_R \left(\frac{a_{\max}}{a_L} - 1\right) + (\rho_R - \rho_g)\left(\frac{a_L}{a_{\max}} - 1\right) \qquad [11]$$

α_{max} being usually much larger than α_L, the second term may become negligible, therefore:

$$Y = \rho_R\left(\frac{a_{\max}}{a_L} - 1\right) \qquad [12]$$

It would be convenient to suppress the factor ρ_R which is difficult to estimate. Let's call V_1 the volume occupied by copper and V_2 the volume of the gangue. The average density ρ of their mixture can be expressed as follows:

$$\rho = \frac{V_1 \cdot \rho_M + V_2 \cdot \rho_g}{V_1 + V_2} \qquad [13]$$

with p_M being the density of the copper mineral.

But $V_1 = \frac{a_L}{\rho_M}$ and $V_2 = \frac{(1-a_L)}{\rho_g}$, then:

$$\rho = \frac{\rho_M \cdot \rho_g}{a_L \cdot \rho_g + \rho_M (1-a_L)} \quad \text{or}$$

$$\frac{\rho_M \cdot \rho_g}{\rho} = a_L \cdot \rho_g + \rho_M (1-a_L) \qquad [14]$$

By transposing [13] into the general equation [7] to calculate the mineralogical factor and after simplifications we obtain:

$$c = \frac{(1-a_L)\rho_M \cdot \rho_g}{a_L \cdot \rho} \qquad [15]$$

therefore:

$$\ell = \frac{Y}{c} = \frac{\rho_R\left(\frac{a_{\max}}{a_L} - 1\right)}{\frac{(1-a_L)\rho_M \cdot \rho_g}{a_L \cdot \rho}} \qquad [16]$$

$$\ell = \frac{(a_{\max} - a_L)\rho_R \cdot \rho}{(1-a_L)\rho_M \cdot \rho_g} \qquad [17]$$

In practice we also know that $p_M > p_R > p > p_g$ therefore:

$$\frac{\rho_R \cdot \rho}{\rho_M \cdot \rho_g} \approx 1 \qquad [18]$$

We finally obtained the very practical formula:

$$\ell = \frac{a_{\max} - a_L}{1 - a_L} \qquad [19]$$

a_{max} and a_L should be expressed as a proportion of the copper mineral content (i.e., as part of one), and not as a metal content. This is the preferred way Pierre Gy always modelled the variability of the liberation factor for various stages of comminution.

3. Recommended method #1: Determination of a_{max} for each size fraction of a typical granulometric distribution

1. Collect a large composite sample representing a single geological unit, from leftover half core samples (e.g., fifty 6-kg samples).
2. Dry the composite.
3. Crush the composite to d = 2.54 cm. The definition of d is the size of a screen retaining no more than 5% of the material by weight.
4. Screen the entire composite using the following screens: 2.54 cm, 1.25 cm, 0.635 cm, 0.335 cm, 0.17 cm, 0.085 cm, 0.0 425cm, and 0.0212 cm. Below 0.0425 cm the method becomes very awkward but it can be done.
5. Wash, dry, and weigh each size fraction.
6. Spread the size fraction between 2.54 cm and 1.25 cm on a clean surface.
7. Using a portable X-ray machine select 10 fragments showing the highest copper content. Using a microscope, identify the main copper mineral to calculate the mineral content. Crush and grind them in a mortar, then assay them for their copper content. You obtain 10 copper results. Look at the distribution of the 10 results. Calculate the average of the 10 results and call the average an estimate of a_{max} for d = 2.09 cm. Using formula [19] calculate ℓ for d = 2.09 cm.
8. Spread the size fraction between 1.25 cm and 0.635 cm on a clean surface.
9. Using a portable X-ray machine select 10 fragments showing the highest copper content. Crush and grind them in a mortar, then assay them for their copper content. You obtain 10 copper results. Look at the distribution of the 10 results. Calculate the average of the 10 results and call the average an estimate of α_{max} for d = 1.05 cm. Using formula [19] calculate ℓ for d = 1.05 cm.
10. Repeat the same process for the other size fractions between 0.635 cm and 0.335 cm, between 0.335 cm and 0.17 cm.

11. For the smaller size fraction, identify 10 zones where the copper content is high with the X-ray machine. At each of these zones collect a spoonful of fragments. Look at them under the microscope and estimate α_{max} using proportion standard references from the mineralogist when you spot a fragment with high copper content. α_{max} will be the average of your observation from the 10 spoonful sub-samples you collected from each respective size fraction. Then, you can calculate ℓ using formula [19] for each respective size fraction.

4. Recommended method #2: Determination of α_{max} for the top size fraction of a typical comminution stage

This method is longer but may be more accurate because of the limitation of hypothesis #2 under different conditions of comminution.

1. Collect a large composite sample representing a single geological unit, from leftover half core samples (e.g., fifty 6-kg samples).
2. Dry the composite.
3. Crush the composite to d = 2.54 cm.
4. Split the composite into 7 sublots.
5. Screen one sublot using 2.54 cm and 1.25 cm screens
6. Wash, dry, and weigh the size fraction.
7. Spread the size fraction between 2.54 cm and 1.25 cm on a clean surface.
8. Using a portable X-ray machine select 10 fragments showing the highest copper content. Using a microscope, identify the main copper mineral to calculate the mineral content. Crush and grind them in a mortar, then assay them for their copper content. You obtain 10 copper results. Look at the distribution of the 10 results. Calculate the average of the 10 results and call the average an estimate of α_{max} for d = 2.09 cm. Using formula [19] calculate ℓ for d = 2.09 cm.
9. Crush the second sublot to d = 1.25 cm
10. Screen the sublot using 1.25 cm and 0.635 cm screens
11. Wash, dry, and weigh the size fraction.
12. Spread the size fraction between 1.25 cm and 0.635 cm on a clean surface.
13. Using a portable X-ray machine select 10 fragments showing the highest copper content. Crush and grind them in a mortar, then assay them for their copper content. You obtain 10 copper results. Look at the distribution of the 10 results. Calculate the average of the 10 results and call the average an estimate of α_{max} for d = 1.05 cm. Using formula [19] calculate ℓ for d = 1.05 cm.
14. Repeat the same process for the other size fractions between 0.635 cm and 0.335 cm, between 0.335 cm and 0.17 cm, by crushing another sublot appropriately each time.
15. For the smaller size fraction, after crushing a sublot appropriately, identify 10 zones where the copper content is high with the X-ray machine. At each of these zones collect a spoonful of fragments. Look at them under the microscope and estimate α_{max} using proportion standard references from the mineralogist. α_{max} will be the average of your observations from the 10 spoonful sub-samples you collected from each respective size fraction. Then, you can calculate ℓ using formula [19] for each respective size fraction.

5. The Empirical Approach Mentioned but Not Used by Pierre Gy

In Gy's earlier literature the *Constant Factor of Constitution Heterogeneity* IH_L was written as follows:

$$IH_L = f \cdot g \cdot c \cdot \ell \cdot d^3 = C \cdot d^3 \qquad [20]$$

The problem with this presentation was that C, which is the product of four factors, must be calculated every time the value of d changes since the liberation factor varies rapidly with the value of d. As a result, in the new literature it became a tradition, for practicality, to summarize the value of IH_L as follows:

$$IH_L = f \cdot g \cdot c \cdot \ell \cdot d^3 = K \cdot d^x \qquad [21]$$

In this new presentation, the liberation factor is assumed to follow an empirical model such as:

$$\ell = \left(\frac{d_\ell}{d}\right)^r \qquad [22]$$

where d_ℓ is defined as the liberation size of the constituent of interest.

In many cases when the constituent of interest is a single mineral, the exponent r is not far away from 0.5. But, as clearly shown by Gy[1] for the liberation of ash in coals, and further demonstrated by François-Bongarçon (2005) for gold, r is not necessarily anywhere close to 0.5, especially when the constituent of interest is located in various minerals.

Under such new conditions, equation [20] should be approximated as follows:

$$s_{FSE}^2 = \left[\frac{1}{M_S} - \frac{1}{M_L}\right] K \cdot d^x \qquad [23]$$

where:

$$K = f \cdot g \cdot c \cdot (d_\ell)^r \qquad [24]$$

and

$$x = 3 - r \qquad [25]$$

K and r then become the key factors to quantify in various experiments described by François-Bongarçon (2005). The author of this paper favors the approach using a_{max} for the determination of the liberation factor, in which case $x = 3$; nevertheless, François-Bongarçon's approach proved to be extremely useful in the mining industry. However, there may be a few problems when we make an attempt to compare oranges and apples, an action that may derail a so-called calibration of Pierre Gy's formula shown by equation [6].

5.1 Problems with the empirical approach

This section is a summary of Pierre Gy's philosophy well explained in his 1967 superb analysis.

The reconciliation of the theoretical approach with the empirical approach is often a complex matter. Such reconciliation is often judged very important to estimate the real value of the theoretical approach. When the difference between the theory and the real observations is too large, we can on the one hand say that the theoretical approach is incomplete or on the other hand say that the experimental observations are affected by other sources of error. It can become very difficult to analyze these annoying differences. For example, if we compare the variance of the Fundamental Sampling Error with the variance obtained from experiments, most of the time this second variance is superior to the first one. The reason is that the Fundamental Sampling Error is not the only error taking place during experiments. The conclusion is that the theoretical approach must be completed by observation from the empirical approach: both domains can complete each other if looked at it with an open mind.

The problem is that many sources of error cannot be quantified, such as variance of GSE, of IDE, of IEE, of IPE, etc. because these errors highly depend on the omnipresence of gravity generating segregation that is a transient phenomenon changing all the time.

Therefore, in the same way that some pure theoreticians refuse to account results from experiments, we may have pure empiricists discouraged by imperfect theories who find solutions with experiments and are in denial about the benefits from the theoretical analysis. Results from the empirical experiments may give us an idea about the global reality of all sources or errors; however, they are incapable of giving us logical avenues to reach all the individual causes of problems; only the theoretical approach can provide acceptable explanations.

So, going back to equations 21 through 25, it is a dangerous approach to mix oranges and apples to modify equation [5], then equation [20] (so called calibration of Gy's formula). Such practice gives a false sense of security when it is tampering with the logical rigor of the theoretical approach.

Following many years of experience the author of this paper rejects this empirical modification using equations [24] and [25] for the following reasons:

1. The sample-tree experiments lead to the cumulated variances of FSE, GSE, IDE, IEE, IPE and even AE giving no solutions for these sources of error and blaming everything on FSE.
2. When d is smaller than 1 cm, results overestimate the variance of FSE, ignoring problem with delayed comminution of the constituent of interest. Indeed, problems due to the delayed comminution of hard minerals and some soft minerals such as gold, other precious metals, molybdenite, native copper, etc. are poorly addressed.
3. When d is larger than 1 cm, results underestimate the variance of FSE, making it very difficult to represent the coarse size fractions in the collected sample. This is a huge problem.
4. It is not clear what the units are for d^{3-r}, which is an important detail.

6. Conclusions

The empirical approach strongly interferes with the integrity of Gy's formulas shown in equations [2] and [6]. As a result, it is wrong to call such approach as the "calibration" of Gy's well-known formula. Furthermore, problems with the empirical approach to calculate the liberation factor and even the theoretical approach not popular and not as well-known for most sampling theory practitioners suggest the use of a completely different approach bypassing the use of the liberation factor.
The introduction of the liberation factor concept was a magnificent academic achievement and will remain as such for those willing to use the theoretical approach. However, for those not comfortable with the current suggestions offered by both the theoretical and empirical approaches, we may suggest the following strategy that will erase any ambiguity.

7. Common Sense Suggestions

The liberation factor was an attempt to include in a single formula requirement to represent fairly all size fractions including the most difficult one to represent which is the coarse size fraction and represent also the particles of the constituent of interest. It is not a must to use the concept of liberation factor if the practitioner follows the two following Cardinal Rules.

7.1 Cardinal Rule #1: Make sure to represent all size fractions.

If the selected sample mass cannot represent all size fractions including the most difficult to represent which is the top size fraction, then the selected sample will not be representative of anything else, including the constituent of interest. Therefore, the necessary sample mass must be calculated using the following general formula suggested by Pierre Gy:

$$s^2_{FSE} = \left[\frac{1}{M_S} - \frac{1}{M_L}\right] f \cdot \rho \left[\left(\frac{1}{a_{Lc}} - 2\right) d^3_{FLc} + \sum_x d^3_{FLx} \cdot a_{Lx}\right] \quad [26]$$

This formula can often be simplified for many applications:

- If $M_L > 10M_S$
- If d_{FLc} is not much different from d
- If a_{Lc} is small, then

$$s^2_{FSE} = \frac{f \cdot \rho}{M_S}\left[\frac{1}{a_{Lc}} - 2\right] d^3_{FLc} \quad [27]$$

If the size fraction of interest is d which is the size opening of a screen retaining no more than 5% of the material, then the final simplified formula is as follows:

$$s^2_{FSE} = \frac{18 \cdot f \cdot \rho \cdot d^3}{M_S} \quad [28]$$

7.2 Cardinal Rule #2: Make sure to represent the coarsest particles of the constituent of interest.

The following Gy's formula putting the emphasis on the constituent of interest alone can be used:

$$s^2_{FSE} = \left[\frac{1}{M_S} - \frac{1}{M_L}\right] \frac{f_M \cdot g_M \cdot \rho_M \cdot d^3_M}{a_L} \quad [29]$$

where (with subscript *M* referring to the Mineral of Interest):

f_M the shape factor of the constituent of interest

g_M the particle size distribution factor of the constituent of interest

ρ_M the density of the constituent of interest

d_M the maximum size of the constituent of interest particle, liberated or not, or cluster of such particles contained in a single fragment of the surrounding matrix; d_M is defined as the size of a screen that would retain no more than 5% by weight of all the particles of the constituent of interest.

Both Cardinal Rules should follow the same Data Quality Objectives (DQO), then the largest necessary sample mass obtained from these two rules must decide what the most appropriate sample mass should be.

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Recommended Recources

The following provide readers with additional historical documents dealing with more depth of the problems listed in this paper:

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